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# M/M/C Bulk Arrival And Bulk Service Queue With Randomly Varying Environment

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***Abstract*: -** This paper studies two stochastic bulk arrival and bulk service C server queues (A) and (B) with k varying environments. The arrival and service times are exponential random variables and their parameters change when the environment changes. The system has infinite storing capacity and the arrival and service sizes are finite valued random variables. Matrix partitioning method is used to study the models. In Model (A) the maximum of the arrival sizes M in all the environments is greater than the maximum of the service sizes N in all the environments, (M > N), and the infinitesimal generator is partitioned as blocks of k times the maximum of the arrival sizes for analysis. In Model (B) the maximum of the arrival sizes M in all the environments is less than the maximum of the service sizes N in all the environments, (M < N), where the generator is partitioned using blocks of k times the maximum of the service sizes. Five different cases associated with C, M and N due to partitions are treated. They are namely, (A1) M >N ≥ C, (A2) M >C >N (A3) C >M >N, which come up in Model (A); (B1) N ≥ C and (B2) C >N, which come up in Model (B) respectively. For the cases when C ≤ M or N Matrix Geometric results are obtained and for the cases when C > both M and N Modified Matrix Geometric results are presented. The basic system generator is seen as a block circulant matrix in all the cases. The stationary queue length probabilities, its expected values, its variances and probabilities of empty queue levels are derived for the models using Matrix Methods. Numerical examples are presented for illustration.

***Keywords*:** *Block Circulant, Bulk Arrival, Bulk Service, C servers, Infinitesimal Generator, Matrix methods*.

## I. INTRODUCTION

In this paper two bulk arrival and bulk service multi server queues have been studied with random environment using matrix partitioning methods. Rama Ganesan, Ramshankar and Ramanarayanan in [1] and [2] have treated M/M/1 bulk queue with random environment and PH/PH/1 bulk queue using Matrix Geometric Methods. Bini, Latouche and Meini [3] have studied numerical methods for Markov chains. Chakravarthy and Neuts [4] have discussed in depth a multi-server queue model. Gaver, Jacobs and Latouche [5] have treated birth and death models with random environment. Latouche and Ramaswami [6] have studied Analytic methods. For matrix geometric methods and models one may refer Neuts [7]. The models considered in this paper are general compared to existing queue models since a multi server bulk arrival and bulk service queue with random environment has not been studied at any depth so far. Here random number of arrivals and random number of services are considered at a time whereas a fixed number of customers arrive or are served at any arrival or service epochs in many existing queue models in literature. Bulk service queue model, with service for fixed b customers when more than b customers are waiting, has been studied by Neuts and Nadarajan [8] for single server system. In the models considered here, depending on the environment the service and arrival sizes are random with distinct discrete distributions and the parameters of exponential distributions of arrival and service times vary. The number of servers increases with number of customers till it becomes C. The model with random arrival is similar to production systems. When a machine manufactures a fixed number of products in every production schedule, the defective items are always rejected in all productions; making any production lot is only of random size and not a fixed one always. Bulk service situations are seen often in software based industries where finished software projects waiting for marketing are sold in bulk sizes when there is economic boom and the business may be very dull when there is economic recession. In industrial productions, bulk types are very common. Manufactured products arrive in bulk sizes and several bulk sizes of products are sold in markets. Recently M/M/1 queue system with disaster has been studied by Noam Paz and Uri Yechali [9] but random arrival size or random service size with varying environments is not studied. Usually the partitions of the bulk arrival models have M/G/1 upper-Heisenberg block matrix structure with zeros below the first sub diagonal. The decomposition of a Toeplitz sub matrix of the infinitesimal generator is required to find the stationary probability vector. Matrix geometric structures have not been noted so far as mentioned by William J. Stewart [10]. But in this paper the partitioning of the matrix is carried out in a way that the stationary probability vectors have a Matrix Geometric solution or a Modified Matrix Geometric solution for infinite capacity C server bulk arrival and bulk service queues with randomly varying environments.

Two models (A) and (B) on M/M/C bulk queue systems under k varying environments with infinite storage space for customers are studied here using the block partitioning method. In the models considered here, the maximum arrival sizes and the maximum service sizes may be different for different environments. Model (A) presents the case when M, the maximum of all the maximum arrival sizes in all the environments is bigger than N, the maximum of all the maximum sale sizes in all the environments. In Model (B), its dual, N is bigger than M, is treated. In general in Queue models, the state space of the system has the first co-ordinate indicating the number of customers in the system but here the customers in the system are grouped and considered as members of M sized blocks of customers (when M >N) or N sized blocks of customers (when N > M) for finding the rate matrix. Using the maximum of the bulk arrival size or maximum of the bulk service size and grouping the customers as members of the blocks for the partitioning the matrix of the infinitesimal generator along with the environment state is a new approach in this area. In [1], Rama Ganesan, Ramshankar and Ramanarayanan for single server M/M/1 bulk queue, have noticed two cases namely M >N and N >M but in this paper because a multi server system is of interest five cases are noticed. Model (A) gives three cases namely (A1) M > N ≥ C, (A2) M > C > N and (A3) C > M > N and Model (B) gives two cases namely (B1) N ≥ C, and (B2) C > N. The case M=N with various C values can be treated using Model (A) or Model (B). For the cases when C ≤ M or N, Matrix Geometric results are obtained and for the cases when C > both M and N, Modified Matrix Geometric results are presented. The matrices appearing as the basic system generators in these models due to block partitions are seen as block circulant matrices. The stationary probability of the number of customers waiting for service, the expected queue length, the variance and the probability of empty queue are derived for these models. Numerical cases are presented to illustrate their applications. The paper is organized in the following manner. In section II and section III the M/M/C bulk arrival and bulk service queues with randomly varying environment in which maximum arrival size M is greater than maximum service size N and the maximum arrival size is M less than the maximum service size N are studied respectively with their sub cases. In section IV numerical cases are presented.

## II. MODEL (A). MAXIMUM ARRIVAL SIZE M IS GREATER THAN MAXIMUM SERVICE SIZE N

### 2.1Assumptions for M > N.

i) There are k environments. The environment changes as per changes in a continuous time Markov chain with infinitesimal generator $Q_1$ of order k with stationary probability vector ϕ.

ii) Customers arrive in different bulk sizes for service. The time between consecutive bulk arrivals of customers has exponential distribution with parameter$\lambda_i$, in the environment i for 1 ≤ i ≤ k. At each bulk arrival in the environment i, $\chi_i$ customers arrive with probability P ($\chi_i$= j) = $p_j^i$ for 1 ≤ j ≤ $M_i$ and $\sum_{j=1}^{M_i} p_j^i$=1 for 1 ≤ i ≤ k.

iii) Customers are served in batches of different bulk sizes. There are s servers to serve when s customers are present in the system for 1≤ s ≤ C. When C or more than C customers are present in the system the number of servers to serve customers is C. In the environment i for1 ≤ i ≤ k, the time between consecutive bulk services has exponential distribution with parameter s$\mu_i$ when s customers are in the system for 1≤ s ≤ C and with parameter C$\mu_i$ when C or more than C customers are present where $\mu_i$ is the parameter of single server exponential service time distribution. At each service epoch in the environment i, $\psi_i$ customers are served with probability given by P ($\psi_i$ = j) = $q_j^i$ for 1≤ j ≤ $N_i$ when more than $N_i$ customers are waiting for service where $\sum_{j=1}^{N_i} q_j^i$ =1. When n customers n < $N_i$ are in the system, then j customers are served with probability, $q_j^i$

for 1≤ j ≤ n-1 and n customers are served with probability $\sum_{j=n}^{N_i} q_j^i$ for 1 ≤ i ≤ k.
iv) When the environment changes from i to j, the parameter of time between consecutive bulk arrivals and the service parameter change from $(\lambda_i, \mu_i)$ $to$ $(\lambda_j, \mu_j)$, the bulk arrival size $\chi_i$ changes to $\chi_j$, the bulk service size $\psi_i$ changes to $\psi_j$ and the maximum arrival size $M_i$ and the maximum service size $N_i$ change to $M_j$ $and$ $N_j$ for 1≤i,j≤k.
v) The maximum of the maximum of arrival sizes M = $\max_{1 \le i \le k} M_i$ is greater than the maximum of the maximum of service sizes N =$\max_{1 \le i \le k} N_i$.

**2.2 Analysis**

There are three sub cases for this model namely (A1) M > N ≥ C, (A2) M > C >N and (A3) C > M >N. Sub Cases (A1) and (A2) admit Matrix Geometric solutions and they are treated in sub section (2.2.1). Modified Matrix Geometric solution is presented for Sub Case (A3) which is studied in sub section (2.2.2). The state of the system of the continuous time Markov chain X (t) under consideration is presented as follows. X (t) = {(n, j, i): for 0 ≤ j ≤ M-1; 1 ≤ i ≤ k and n ≥ 0}
(1) The chain is in the state (n, j, i) when the number of customers in the system is n M + j, for 0 ≤ j ≤ M-1 and 0 ≤ n < ∞ and the environment is i for 1 ≤ i ≤ k. When the number of customers in the system is r, then r is identified with (n, j) where r on division by M gives n as the quotient and j as the remainder. Let the survivor probability of the number of arrivals at an arrival epoch and the number of services at a service epoch in the environment i for 1 ≤ i ≤ k be P ($\chi_i$ > j) = $P_j^i$ =1-$\sum_{n=1}^{j} p_n^i$, and $P_0^i = 1$ for 1 ≤ j ≤ $M_i$ -1 (2)
and P ($\psi_i$ > j) = $Q_j^i$ =1-$\sum_{n=1}^{j} q_n^i$, and $Q_0^i = 1$ for 1 ≤ j ≤ $N_i$ -1 (3)

**2.2.1 Sub Cases: (A1) M > N ≥ C and (A2) M > C > N**

When M > N ≥ C or M > C > N, the M/M/C bulk queue admits matrix geometric solution as follows. The chain X (t) describing them, has the infinitesimal generator $Q_{A,2.1}$ of infinite order which can be presented in block partitioned form given below.

$$Q_{A,2.1} = \begin{bmatrix} B_1 & A_0 & 0 & 0 & . & . & . & \cdots \\ A_2 & A_1 & A_0 & 0 & . & . & . & \cdots \\ 0 & A_2 & A_1 & A_0 & 0 & . & . & \cdots \\ 0 & 0 & A_2 & A_1 & A_0 & 0 & . & \cdots \\ 0 & 0 & 0 & A_2 & A_1 & A_0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \quad (4)$$

In (4) the states of the matrices are listed lexicographically as $\underline{0}, \underline{1}, \underline{2}, \underline{3}, \ldots. \underline{n}, \ldots$. Here the vector $\underline{n}$ is of type 1 x k M and $\underline{n}$ = ((n, 0, 1),(n, 0, 2)…(n, 0, k),(n, 1, 1),(n, 1, 2)…(n, 1, k)…(n, M-1, 1),(n, M-1, 2)…(n, M-1, k)) for n ≥ 0. The matrices $B_1 \, and \, A_1$ have negative diagonal elements, they are of order Mk and their off diagonal elements are non- negative. The matrices $A_0, and A_2$ have nonnegative elements and are of order Mk and they are given below. Let the following be diagonal matrices of order k

$\Lambda_j$ =diag $(\lambda_1 p_j^1, \lambda_2 p_j^2, \ldots, \lambda_k p_j^k)$ $for\ 1 \le j \le M;$ $U_j$ = diag $(\mu_1 q_j^1, \mu_2 q_j^2, \ldots, \mu_k q_j^k)$ $for\ 1 \le j \le N$ (5)

$V_j$ = diag $(\mu_1 Q_j^1, \mu_2 Q_j^2, \ldots, \mu_k Q_j^k)$ $for\ 1 \le j \le N;$ $\Lambda$ =diag $(\lambda_1, \lambda_2, \ldots, \lambda_k)$; $U$ = diag $(\mu_1, \mu_2, \ldots, \mu_k)$ (6)

$Let\ Q_1' = Q_1 - \Lambda - CU.$ (7)

Here $Q_1$ is the infinitesimal generator of the Markov chain of the environment defined earlier

$$A_0 = \begin{bmatrix} \Lambda_M & 0 & \cdots & 0 & 0 & 0 \\ \Lambda_{M-1} & \Lambda_M & \cdots & 0 & 0 & 0 \\ \Lambda_{M-2} & \Lambda_{M-1} & \cdots & 0 & 0 & 0 \\ \Lambda_{M-3} & \Lambda_{M-2} & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \Lambda_3 & \Lambda_4 & \cdots & \Lambda_M & 0 & 0 \\ \Lambda_2 & \Lambda_3 & \cdots & \Lambda_{M-1} & \Lambda_M & 0 \\ \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-2} & \Lambda_{M-1} & \Lambda_M \end{bmatrix} (8) \qquad A_2 = \begin{bmatrix} 0 & \cdots & 0 & CU_N & CU_{N-1} & \cdots & CU_2 & CU_1 \\ 0 & \cdots & 0 & 0 & CU_N & \cdots & CU_3 & CU_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & CU_N & CU_{N-1} \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & CU_N \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} (9)$$

$$A_1 = \begin{bmatrix} Q_1' & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-N-2} & \Lambda_{M-N-1} & \Lambda_{M-N} & \cdots & \Lambda_{M-2} & \Lambda_{M-1} \\ CU_1 & Q_1' & \Lambda_1 & \cdots & \Lambda_{M-N-3} & \Lambda_{M-N-2} & \Lambda_{M-N-1} & \cdots & \Lambda_{M-3} & \Lambda_{M-2} \\ CU_2 & CU_1 & Q_1' & \cdots & \Lambda_{M-N-4} & \Lambda_{M-N-3} & \Lambda_{M-N-2} & \cdots & \Lambda_{M-4} & \Lambda_{M-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ CU_N & CU_{N-1} & CU_{N-2} & \cdots & Q_1' & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-N-2} & \Lambda_{M-N-1} \\ 0 & CU_N & CU_{N-1} & \cdots & CU_1 & Q_1' & \Lambda_1 & \cdots & \Lambda_{M-N-3} & \Lambda_{M-N-2} \\ 0 & 0 & CU_N & \cdots & CU_2 & CU_1 & Q_1' & \cdots & \Lambda_{M-N-4} & \Lambda_{M-N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & CU_N & CU_{N-1} & CU_{N-2} & \cdots & Q_1' & \Lambda_1 \\ 0 & 0 & 0 & \cdots & 0 & CU_N & CU_{N-1} & \cdots & CU_1 & Q_1' \end{bmatrix} \quad (10)$$

The matrix $B_1$ for Sub Case (A1) where N > C and Sub Case (A2) where C > N are given below in (11) and (12) respectively. For the case when C=N, the matrix $B_1$ may be written by placing C in place of N in the N-th block row in (12) and there after the multiplier of $U_j$ is C. Let $Q'_{1,j} = Q_1 - \Lambda - jU$ for $0 \leq j \leq C$ and $Q'_{1,C} = Q'_1$

$$B_1 = \begin{bmatrix} Q'_{1,0} & \Lambda_1 & \Lambda_2 & \Lambda_3 & \cdots & \Lambda_C & \Lambda_{C+1} & \Lambda_{C+2} & \cdots & \Lambda_N & . & \cdots & \Lambda_{M-N-2} & \Lambda_{M-N-1} & \cdots & \Lambda_{M-2} & \Lambda_{M-1} \\ U & Q'_{1,1} & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{C-1} & \Lambda_C & \Lambda_{C+1} & \cdots & \Lambda_{N-1} & . & \cdots & . & . & \cdots & \Lambda_{M-3} & \Lambda_{M-2} \\ 2V_1 & 2U_1 & Q'_{1,2} & \Lambda_1 & \cdots & \Lambda_{C-2} & \Lambda_{C-1} & \Lambda_C & \cdots & \Lambda_{N-2} & . & \cdots & . & . & \cdots & \Lambda_{M-4} & \Lambda_{M-3} \\ 3V_2 & 3U_2 & 3U_1 & Q'_{1,3} & \cdots & \Lambda_{C-3} & \Lambda_{C-2} & \Lambda_{C-1} & \cdots & \Lambda_{N-3} & . & \cdots & . & . & \ddots & \Lambda_{M-5} & \Lambda_{M-4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ CV_{C-1} & CU_{C-1} & CU_{C-2} & CU_{C-3} & \cdots & Q'_{1,C} & \Lambda_1 & \Lambda_2 & \cdots & . & . & \cdots & . & . & \cdots & \Lambda_{M-C-2} & \Lambda_{M-C-1} \\ CV_C & CU_C & CU_{C-1} & CU_{C-2} & \cdots & CU_1 & Q'_1 & \Lambda_1 & \cdots & . & . & \cdots & . & . & \cdots & \Lambda_{M-C-3} & \Lambda_{M-C-2} \\ CV_{C+1} & CU_{C+1} & CU_C & CU_{C-1} & \cdots & CU_2 & CU_1 & Q'_1 & \cdots & . & . & \cdots & . & . & \cdots & \Lambda_{M-C-4} & \Lambda_{M-C-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ CV_{N-1} & CU_{N-1} & CU_{N-2} & CU_{N-3} & \cdots & CU_{N-C} & CU_{N-C-1} & CU_{N-C-2} & \cdots & Q'_1 & \Lambda_1 & \cdots & \Lambda_{M-2N-2} & \Lambda_{M-2N-1} & \cdots & \Lambda_{M-N-2} & \Lambda_{M-N-1} \\ 0 & CU_N & CU_{N-1} & CU_{N-2} & \cdots & CU_{N-C+1} & CU_{N-C} & CU_{N-C-1} & \cdots & CU_1 & Q'_1 & \cdots & . & . & \cdots & \Lambda_{M-N-3} & \Lambda_{M-N-2} \\ 0 & 0 & CU_N & CU_{N-1} & \cdots & CU_{N-C+2} & CU_{N-C+1} & CU_{N-C} & \cdots & CU_2 & CU_1 & \cdots & . & . & \cdots & \Lambda_{M-N-4} & \Lambda_{M-N-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & CU_N & CU_{N-1} & \cdots & Q'_1 & \Lambda_1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & CU_N & \cdots & CU_1 & Q'_1 \end{bmatrix} \quad (11)$$

$$B_1 = \begin{bmatrix} Q'_{1,0} & \Lambda_1 & \Lambda_2 & \Lambda_3 & \cdots & \Lambda_N & \Lambda_{N+1} & \cdots & \Lambda_C & . & \cdots & . & . & \cdots & \Lambda_{M-2} & \Lambda_{M-1} \\ U & Q'_{1,1} & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{N-1} & \Lambda_N & \cdots & \Lambda_{C-1} & . & \cdots & . & . & \cdots & \Lambda_{M-3} & \Lambda_{M-2} \\ 2V_1 & 2U_1 & Q'_{1,2} & \Lambda_1 & \cdots & \Lambda_{N-2} & \Lambda_{N-1} & \cdots & \Lambda_{C-2} & . & \cdots & . & . & \cdots & \Lambda_{M-4} & \Lambda_{M-3} \\ 3V_2 & 3U_2 & 3U_1 & Q'_{1,3} & \cdots & \Lambda_{N-3} & \Lambda_{N-2} & \cdots & \Lambda_{C-3} & . & \cdots & . & . & \ddots & \Lambda_{M-5} & \Lambda_{M-4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ NV_{N-1} & NU_{N-1} & NU_{N-2} & NU_{N-3} & \cdots & Q'_{1,N} & \Lambda_1 & \cdots & . & . & \cdots & . & . & \cdots & . & . \\ 0 & (N+1)U_N & (N+1)U_{N-1} & (N+1)U_{N-2} & \cdots & (N+1)U_1 & Q'_{1,N+1} & \cdots & . & . & \cdots & . & . & \cdots & . & . \\ 0 & 0 & (N+2)U_N & (N+2)U_{N-1} & \cdots & (N+2)U_2 & (N+2)U_1 & \cdots & . & . & \cdots & . & . & \cdots & . & . \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & CU_N & CU_{N-1} & \cdots & Q'_{1,C} & \Lambda_1 & \cdots & . & . & \cdots & . & . \\ 0 & 0 & 0 & 0 & \cdots & 0 & CU_N & \cdots & CU_1 & Q'_1 & \cdots & . & . & \cdots & . & . \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & CU_N & CU_{N-1} & \cdots & Q'_1 & \Lambda_1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & CU_N & \cdots & CU_1 & Q'_1 \end{bmatrix} \quad (12)$$

The basic generator $Q''_A$ of the bulk queue, which is concerned with only the arrival and the service, is a matrix of order Mk given above in (13) where $Q''_A = A_0 + A_1 + A_2$. It is well known that a square matrix in which each row (after the first) has the elements of the previous row shifted cyclically one place right, is called a circulant matrix. It is very interesting to note that the matrix $Q''_A$ is a block circulant matrix where each block matrix is rotated one block to the right relative to the preceding block partition.

$$Q''_A = \begin{bmatrix} Q'_1+\Lambda_M & \Lambda_1 & \cdots & \Lambda_{M-N-2} & \Lambda_{M-N-1} & \Lambda_{M-N}+CU_N & \cdots & \Lambda_{M-2}+CU_2 & \Lambda_{M-1}+CU_1 \\ \Lambda_{M-1}+CU_1 & Q'_1+\Lambda_M & \cdots & \Lambda_{M-N-3} & \Lambda_{M-N-2} & \Lambda_{M-N-1} & \cdots & \Lambda_{M-3}+CU_3 & \Lambda_{M-2}+CU_2 \\ \Lambda_{M-2}+CU_2 & \Lambda_{M-1}+CU_1 & \cdots & \Lambda_{M-N-4} & \Lambda_{M-N-3} & \Lambda_{M-N-2} & \cdots & \Lambda_{M-4}+CU_3 & \Lambda_{M-3}+CU_3 \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots \\ \Lambda_{M-N+2}+CU_{N-2} & . & \cdots & . & . & . & \cdots & \Lambda_{M-N}+CU_N & \Lambda_{M-N+1}+CU_{N-1} \\ \Lambda_{M-N+1}+CU_{N-1} & . & \cdots & . & . & . & . & \Lambda_{M-N-1} & \Lambda_{M-N}+CU_N \\ \Lambda_{M-N}+CU_N & . & \cdots & Q'_1+\Lambda_M & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-N-2} & \Lambda_{M-N-1} \\ \Lambda_{M-N-1} & \Lambda_{M-N}+CU_N & \cdots & \Lambda_{M-1}+CU_1 & Q'_1+\Lambda_M & \Lambda_1 & \cdots & \Lambda_{M-N-3} & \Lambda_{M-N-2} \\ \Lambda_{M-N-2} & \Lambda_{M-N-1} & \cdots & \Lambda_{M-2}+CU_2 & \Lambda_{M-1}+CU_1 & Q'_1+\Lambda_M & \cdots & \Lambda_{M-N-4} & \Lambda_{M-N-3} \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots \\ \Lambda_2 & \Lambda_3 & \cdots & \Lambda_{M-N}+CU_N & \Lambda_{M-N+1}+CU_{N-1} & \Lambda_{M-N+2}+CU_{N-2} & \cdots & Q'_1+\Lambda_M & \Lambda_1 \\ \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-N-1} & \Lambda_{M-N}+CU_N & \Lambda_{M-N+1}+CU_{N-1} & \cdots & \Lambda_{M-1}+CU_1 & Q'_1+\Lambda_M \end{bmatrix} \quad (13)$$

The probability vector w of (13) gives, $wQ_A^{''}$ =0 and w.e = 1. (14)
Since the probability vector of the environment generator $Q_1$ is ϕ, the following are seen $\phi Q_1$ = 0 and ϕ e =1. It can be seen in (13) that the first block-row of type k x Mk is $W = (Q_1' + \Lambda_M, \Lambda_1, \Lambda_2, \ldots \Lambda_{M-N-2}, \Lambda_{M-N-1}, \Lambda_{M-N} + CU_N, \ldots \Lambda_{M-2} + CU_2, \Lambda_{M-1} + CU_1)$ This gives as the sum of the blocks$(Q_1' + \Lambda M + \Lambda 1 + \Lambda 2 + \ldots + \Lambda M-N-2 + \Lambda M-N-1 + \Lambda M-N + CUN + \ldots \ldots + \Lambda M-2 + CU2 + \Lambda M-1 + CU1 = Q1$. Since $\phi Q_1$=0, this gives $\phi(Q_1' + \Lambda_M) + \phi \sum_{i=1}^{M-N-1} \Lambda_i + \phi \sum_{i=1}^{N} (\Lambda_{M-i} + CU_i)$ = 0 which implies (ϕ, ϕ … ϕ, ϕ). W = 0 = (ϕ, ϕ … ϕ, ϕ ) W' where W' is the transpose column vector of W. Since all blocks, in any block-row are seen somewhere in each and every column block due to block circulant structure, the above equation shows the left eigen vector of the matrix $Q_A^{''}$ is (ϕ, ϕ … ϕ). Using (14) $w = \left(\frac{\phi}{M}, \frac{\phi}{M}, \frac{\phi}{M}, \ldots, \frac{\phi}{M}\right)$ (15)
Neuts [7], gives the stability condition as, w $A_0\, e < w\, A_2\, e$ where w is given by (15). Taking the sum of the same cross diagonally using the structure in (8) and (9) for the $A_0$ *and* $A_2$ matrices, it can be seen that w $A_0\, e = \frac{1}{M} \phi \left(\sum_{n=1}^{M} n\Lambda_n\right) e = \frac{1}{M} \phi . (\lambda_1 E(\chi_1), \lambda_2 E(\chi_2), \ldots, \lambda_k E(\chi_k)) < w\, A_2\, e = \frac{1}{M} C\, \phi \left(\sum_{n=1}^{N} nU_n\right) e = \frac{1}{M} C\, \phi . (\mu_1 E(\psi_1), \mu_2 E(\psi_2), \ldots, \mu_k E(\psi_k))$ .
Taking the probability vector of the environment generator $Q_1$ as $\phi = (\phi_1, \phi_2, \ldots, \phi_{k-1}, \phi_k)$ , the inequality reduces to $\sum_{i=1}^{k} \phi_i\, \lambda_i E(\chi_i) < C \sum_{i=1}^{k} \phi_i\, \mu_i E(\psi_i)$. (16)
This is the stability condition for the M/M/C bulk arrival, bulk service queue with random environment for Sub Case (A1) M > N ≥ C and Sub Case (A2) M > C > N. When (16) is satisfied, the stationary distribution exists as proved in Neuts [7]. Let π (n, j, i), for 0 ≤ j ≤ M-1, 1 ≤ i ≤ k and 0 ≤ n < ∞ be the stationary probability of the states in (1) and $\pi_n$ be the vector of type 1xMk with, $\pi_n$= (π(n, 0, 1), π(n, 0, 2) … π(n, 0, k), π(n, 1, 1), π(n, 1, 2),… π(n, 1, k)…π(n, M-1, 1), π(n, M-1, 2)…π(n, M-1, k) )for n ≥ 0. The stationary probability vector $\pi = (\pi_0, \pi_1, \pi_3, \ldots\ldots)$ satisfies $\pi Q_{A,2.1}$=0 and $\pi$e=1. (17)
From (17), it can be seen $\pi_0 B_1 + \pi_1 A_2$=0. (18)
$\pi_{n-1} A_0 + \pi_n A_1 + \pi_{n+1} A_2 = 0$, for n ≥ 1. (19)
Introducing the rate matrix R as the minimal non-negative solution of the non-linear matrix equation $A_0$+R$A_1$+$R^2 A_2$=0, (20)
it can be proved (Neuts [7]) that $\pi_n$ satisfies $\pi_n = \pi_0 R^n$ for n ≥ 1. (21)
Using (18) and (21), $\pi_0$ satisfies $\pi_0\, [B_1 + RA_2] = 0$ (22)
Now $\pi_0$ can be calculated up to multiplicative constant by (22). From (17) and (21) $\pi_0\, (I - R)^{-1} e$ =1. (23)
Replacing the first column of the matrix multiplier of $\pi_0$ in equation (22) by the column vector multiplier of $\pi_0$ in (23), a matrix which is invertible may be obtained. The first row of the inverse of that same matrix is $\pi_0$ and this gives along with (21) all the stationary probabilities. The matrix R given in (20) is computed using recurrence relation $R(0) = 0$; $R(n+1) = -A_0 A_1^{-1} - R^2(n) A_2 A_1^{-1}$ , n ≥ 0. (24)
The iteration may be terminated to get a solution of R at an approximate level where $\|R(n+1) - R(n)\| < \varepsilon$

**Note:**

The partition of the infinitesimal generator for the case M = C is similar and in that case C does not appear as a multiplier for the $U_j$ matrices in the matrix $B_1$ (11) and (12) in the $\underline{0}$ block of (4) and C appears as a multiplier for all $U_j$ matrices in the matrices of $A_1$ and $A_2$ from row block $\underline{1}$ on wards. From the arguments presented earlier it can be seen that the system admits Matrix Geometric solution for C = M also.

**2.2.2 Sub Case: (A3) C > M > N**

When C > M > N, the M/M/C bulk queue admits a modified matrix geometric solution as follows. The chain X (t) describing this Sub Case (A3), can be defined as in (1) presented for Sub Cases (A1) and (A2). It has the infinitesimal generator $Q_{A,2.2}$ of infinite order which can be presented in block partitioned form given below. When C > M, let C = m* M + n* where m* is positive integer and n* is nonnegative integer with 0 ≤ n* ≤ M-1.

$$Q_{A,2.2}=\begin{bmatrix} B'_1 & A_0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ A_{2,1} & A_{1,1} & A_0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ 0 & A_{2,2} & A_{1,2} & A_0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & A_{2,3} & A_{1,3} & A_0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & A_{2,m*} & A_{1,m*} & A_0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & A_2 & A_1 & A_0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & A_2 & A_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \vdots & \ddots \end{bmatrix} \tag{25}$$

In (25) the states of the matrices are listed lexicographically as $\underline{0}, \underline{1}, \underline{2}, \underline{3}, \ldots . \underline{n}, \ldots$. Here the vector $\underline{n}$ is of type 1 x k M and $\underline{n}$ = ((n, 0, 1),(n, 0, 2)…(n, 0, k),(n, 1, 1),(n, 1, 2)…(n, 1, k)…(n, M-1, 1),(n, M-1, 2)…(n, M-1, k)) for n ≥ 0.The matrices $B'_1 \, and \, A_1$ have negative diagonal elements, they are of order Mk and their off diagonal elements are non- negative. The matrices $A_0 \, and A_2$ have nonnegative elements and are of order Mk and the matrices $A_0, A_1 and \, A_2$ are same as defined earlier for Sub Cases (A1) and (A2) in equations (8), (9) and (10). Since C > M the number of servers in the system s equals the number of customers in the system L up to customer length C. When the number of customers L becomes more than C, (L ≥ C), the number of servers in the system becomes constant C. When the number of customers L becomes less than C (L<C), the number of servers reduces and equals the number of customers. The matrix $A_{2,j}$ for 1 ≤ j < m*-1 is given below

$$A_{2,j} = \begin{bmatrix} 0 & \cdots & 0 & jMU_N & jMU_{N-1} & \cdots & jMU_2 & jMU_1 \\ 0 & \cdots & 0 & 0 & (jM+1)U_N & \cdots & (jM+1)U_3 & (jM+1)U_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & (jM+N-2)U_N & (jM+N-2)U_{N-1} \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & (jM+N-1)U_N \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{26}$$

The matrix $A_{2,m*}$ is as follows given in (27) when C = m*M + n* and n* is such that 0 ≤ n* ≤ N-1. Here the multiplier of $U_j$ in the row block increases by one till the multiplier becomes C = m*M + n* and there after the multiplier is C for $U_j$ for all blocks.

When N ≤ n* ≤ M-1, $A_{2,m*}$ is same as in (26) for j = m*.

$$A_{2,m*} = \begin{bmatrix} 0 & \cdots & 0 & (Mm*)U_N & (Mm*)U_{N-1} & \cdots & . & \cdots & (Mm*)U_2 & (Mm*)U_1 \\ 0 & \cdots & 0 & 0 & (Mm*+1)U_N & \cdots & . & \cdots & (Mm*+1)U_3 & (Mm*+1)U_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & CU_N & \cdots & CU_{n*+2} & CU_{n*+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & CU_N & CU_{N-1} \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & CU_N \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{27}$$

$$B'_1=\begin{bmatrix} Q'_{1,0} & \Lambda_1 & \Lambda_2 & \Lambda_3 & \cdots & \Lambda_{N+1} & \cdots & \cdot & \cdot & \cdots & \Lambda_{M-2} & \Lambda_{M-1} \\ U & Q'_{1,1} & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_N & \cdots & \cdot & \cdot & \cdots & \Lambda_{M-3} & \Lambda_{M-2} \\ 2V_1 & 2U_1 & Q'_{1,2} & \Lambda_1 & \cdots & \Lambda_{N-1} & \cdots & \cdot & \cdot & \cdots & \Lambda_{M-4} & \Lambda_{M-3} \\ 3V_2 & 3U_2 & 3U_1 & Q'_{1,3} & \cdots & \Lambda_{N-2} & \cdots & \cdot & \cdot & \ddots & \Lambda_{M-5} & \Lambda_{M-4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ NV_{N-1} & NU_{N-1} & NU_{N-2} & NU_{N-3} & \cdots & \Lambda_1 & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & (N+1)U_N & (N+1)U_{N-1} & (N+1)U_{N-2} & \cdots & Q'_{1,N+1} & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & (N+2)U_N & (N+2)U_{N-1} & \cdots & (N+2)U_1 & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & (M-2)U_N & (M-2)U_{N-1} & \cdots & Q'_{1,M-2} & \Lambda_1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & (M-1)U_N & \cdots & (M-1)U_1 & Q'_{1,M-1} \end{bmatrix} \quad (28)$$

Here, $Q'_{1,j}=Q_1-\Lambda-jU$ for $0\le j\le C$ and $Q'_{1,C}=Q'_1$. The matrix $A_{1,j}$ for $1\le j\le m^*-1$ is as follows.

$$A_{1,j}=\begin{bmatrix} Q'_{1,jM} & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-N-2} & \Lambda_{M-N-1} & \cdots & \Lambda_{M-2} & \Lambda_{M-1} \\ (jM+1)U_1 & Q'_{1,jM+1} & \Lambda_1 & \cdots & \Lambda_{M-N-3} & \Lambda_{M-N-2} & \cdots & \Lambda_{M-3} & \Lambda_{M-2} \\ (jM+2)U_2 & (jM+2)U_1 & Q'_{1,jM+2} & \cdots & \Lambda_{M-N-4} & \Lambda_{M-N-3} & \cdots & \Lambda_{M-4} & \Lambda_{M-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (jM+N)U_N & (jM+N)U_{N-1} & (jM+N)U_{N-2} & \cdots & Q'_{1,jM+N} & \Lambda_1 & \cdots & \Lambda_{M-N-2} & \Lambda_{M-N-1} \\ 0 & (jM+N+1)U_N & (jM+N+1)U_{N-1} & \cdots & (jM+N+1)U_1 & Q'_{1,jM+N+1} & \cdots & \Lambda_{M-N-3} & \Lambda_{M-N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (jM+M-2)U_N & (jM+M-2)U_{N-1} & \cdots & Q'_{1,jM+M-2} & \Lambda_1 \\ 0 & 0 & 0 & \cdots & 0 & (jM+M-1)U_N & \cdots & (jM+M-1)U_1 & Q'_{1,jM+M-1} \end{bmatrix} \quad (29)$$

The matrix $A_{1,m*}$ is as follows when C = m*M + n* and n* is such that 0 ≤ n* ≤ N-1. The multiplier of $U_j$ increases by one till it becomes C = m*M + n* and thereafter in all the blocks the multiplier of $U_j$ is C.

$$A_{1,m*}=\begin{bmatrix} Q'_{1,Mm*} & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-N-2} & \Lambda_{M-N-1} & \cdots & \Lambda_{M-2} & \Lambda_{M-1} \\ (Mm*+1)U_1 & Q'_{1,jm*+1} & \Lambda_1 & \cdots & \Lambda_{M-N-3} & \Lambda_{M-N-2} & \cdots & \Lambda_{M-3} & \Lambda_{M-2} \\ (Mm*+2)U_2 & (jm*+2)U_1 & Q'_{1,jm*+2} & \cdots & \Lambda_{M-N-4} & \Lambda_{M-N-3} & \cdots & \Lambda_{M-4} & \Lambda_{M-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (Mm*+n*-1)U_{n*-1} & (jm*+n*-1)U_{n*-2} & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\ CU_{n*} & CU_{n*-1} & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ CU_N & CU_{N-1} & CU_{N-2} & \cdots & Q'_{1,C} & \Lambda_1 & \cdots & \Lambda_{M-N-2} & \Lambda_{M-N-1} \\ 0 & CU_N & CU_{N-1} & \cdots & CU_1 & Q'_{1,C} & \cdots & \Lambda_{M-N-3} & \Lambda_{M-N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & CU_N & CU_{N-1} & \cdots & Q'_{1,C} & \Lambda_1 \\ 0 & 0 & 0 & \cdots & 0 & CU_N & \cdots & CU_1 & Q'_{1,C} \end{bmatrix} \quad (30)$$

When n* = N or n* > N then, in the matrix $A_{1,m*}$, there is slight change in the elements. When n* = N, in the N+1 block row and thereafter C appears as multiplier of $U_j$, and when n* > N with n* = N + r for 1 ≤ r ≤ M-N-1, in the n*+1 block row $U_N$ appears in the r + 1 column block. C appears as multiplier for it and as the multiplier of $U_j$ thereafter in all row blocks respectively. The basic system generator for this Sub Case is same as (13) with probability vector as given in (15). The stability condition is as presented in (16). Once the stability condition is satisfied the stationary probability vector exists by Neuts [7]. As in the previous Sub Cases, $\pi Q_{A,2.2}=0$ and $\pi e=1$. (31)

The following may be noted. $\pi_n A_0+\pi_{n+1}A_1+\pi_{n+2}A_2=0$, for n ≥ m*, the rate matrix R is same as in previous Sub Cases with same iterative method for solving the same and $\pi_n$ satisfies $\pi_n=\pi_{m*}R^{n-m*}$ for n ≥ m*. (32)

The set of equations available from (31) are $\pi_0 B'_1+\pi_1 A_{2,1}=0$, (33)

$\pi_i A_0+\pi_{i+1}A_{1,i+1}+\pi_{i+2}A_{2,i+2}=0$, for $0\le i\le m^*-2$ (34)

and $\pi_{m*-1}A_0+\pi_{m*}A_{1,m*}+\pi_{m*+1}A_2=0$. (35)

The equation $\pi e=1$ in (31) gives $\sum_{i=0}^{m*-1}\pi_i e+\pi_{m*}(I-R)^{-1}e=1$ (36)

Using $\pi_{m*+1}=\pi_{m*}R$ and equations (33), (34), (35) and (36) the following matrix equations can be seen.

$$(\pi_0,\pi_1,\pi_3,\ldots\ldots\pi_{m*})\,Q'_{A,2.2}=0 \quad (37)$$

$$(\pi_0,\pi_1,\pi_3,\ldots\ldots\pi_{m*})\begin{bmatrix} e \\ (I-R)^{-1}e\end{bmatrix}=1 \quad (38)$$

The matrix $Q'_{A,2.2}$ is given by (39).

$$Q'_{A,2.2}=\begin{bmatrix} B'_1 & A_0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ A_{2,1} & A_{1,1} & A_0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & A_{2,2} & A_{1,2} & A_0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & A_{2,3} & A_{1,3} & A_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & A_{2,m*} & RA_2 + A_{1,m*} \end{bmatrix} \quad (39)$$

Equations (37) and (38) may be used for finding $(\pi_0, \pi_1, \pi_3, \ldots \ldots \pi_{m*})$. Replacing the first column of the first column- block in the matrix given by (39) by the column vector multiplier in (38) a matrix which is invertible can be obtained. The first row of the inverse matrix gives $(\pi_0, \pi_1, \pi_3, \ldots \ldots \pi_{m*})$. This together with equation (32) gives all the probability vectors for this Sub Case.

***2.3. Performance Measures***

(1) The probability P(S = r), of the queue length S = r, can be seen as follows. Let m ≥ 0 and n for 0 ≤ n ≤ M-1 be non-negative integers such that r = mM+n. Then it is noted that P (S=r) $=\sum_{i=1}^{k} \pi\ (m, n, i\ )$, where r = m M + n.

(2) P (Queue length is 0) = P (S=0) = $\sum_{i=1}^{k} \pi(0, 0, i)$.

(3) The expected queue level E(S), can be calculated as follows. For Sub Cases (A1) and (A2) it may be seen as follows. Since π (n, j, i) = P [S = M n + j, and environment state = i], for n≥0, and 0 ≤ j ≤ M-1 and 1 ≤ i ≤ k, E(S) $=\sum_{n=0}^{\infty} \sum_{j=0}^{M-1} \sum_{i=1}^{k} \pi(n, j, i)\ (Mn + j)$ $=\sum_{n=0}^{\infty} \pi_n$. (Mn… Mn, Mn+1… Mn+1, Mn+2…Mn+2… Mn+M-1… Mn+M-1) where in the multiplier vector Mn appears k times, Mn+1 appears k times and so on and finally Mn+M-1appears k times. So E(S) $=M\sum_{n=0}^{\infty} n\pi_n\, e\ + \pi_0 (\ I - R)^{-1} \xi$ . Here Mk x1 column vector $\xi = (0, \ldots 0, 1, \ldots, 1, 2, \ldots, 2, \ldots, M-1, \ldots, M-1)'$ where the numbers 0, 1, 2, 3,… and M-1 appear k times in order. This gives

$$E\ (S) = \quad \pi_0 (\ I - R)^{-1} \xi + M\pi_0 (I - R\ )^{-2} Re \quad (40)$$

For Sub Case (A3), E(S) $=\sum_{n=0}^{\infty} \sum_{j=0}^{M-1} \sum_{i=1}^{k} \pi(n, j, i)\ (Mn + j) = M\sum_{n=0}^{\infty} n\pi_n\, e + \sum_{n=0}^{\infty} \pi_n\, \xi = M\sum_{n=0}^{\infty} n\pi_n\, e + \sum_{i=0}^{m*-1} \pi_i \xi + \pi_{m*}(I-R)^{-1}\xi$. Letting the generating function of probability vector $\Phi(s) = \sum_{i=0}^{\infty} \pi_i s^i$, it can be seen, $\Phi(s) = \sum_{i=0}^{m*-1} \pi_i s^i + \pi_{m*}\ s^{m*}(I-Rs)^{-1}$ and $\sum_{n=0}^{\infty} n\pi_n\, e = \Phi'(1)e = \sum_{i=0}^{m*-1} i\pi_i\, e + \pi_{m*} m^*(I-R)^{-1}e + \pi_{m*}(I-R)^{-2}$ Re. This gives

$$E(S) = M\ [\sum_{i=0}^{m*-1} i\pi_i\, e + \pi_{m*} m^*(I-R)^{-1}e \ + \pi_{m*}(I-R)^{-2}\ Re] + \sum_{i=0}^{m*-1} \pi_i \xi\ + \pi_{m*}(I-R)^{-1}\xi \quad (41)$$

(4) Variance of queue level can be seen using Var (S) = E $(S^2)$ – E(S)$^2$. Let η be column vector η = $[0, ..0, 1^2, \ldots 1^2\ 2^2, ..2^2, \ldots\ (M-1)^2, \ldots\ (M-1)^2]'$ of type Mkx1where the squares of the numbers 0,1,2…(M-1) appear k times each in order. Then it can be seen that the second moment, for Sub Cases (A1) and (A2) E $(S^2) = \sum_{n=0}^{\infty} \sum_{j=0}^{M-1} \sum_{i=1}^{k} \pi(n, j, i)\ [M\,n + j]^2 \quad = M^2[\sum_{n=1}^{\infty} n(n-1)\pi_n\, e + \sum_{n=0}^{\infty} n\pi_n\, e] + \sum_{n=0}^{\infty} \pi_n \eta + 2M \sum_{n=0}^{\infty} n\, \pi_n\, \xi$.

$$\text{So,} \quad E(S^2) = M^2[\pi_0(I - R)^{-3} 2R^2\, e + \pi_0(I - R)^{-2} Re] + \pi_0(I - R)^{-1}\eta + 2M\, \pi_0(I - R)^{-2} R\xi \quad (42)$$

Using (40) and (42) the variance can be written for Sub Cases (A1) and (A2). For the Sub Case (A3) the second moment can be seen as follows. E $(S^2) = \sum_{n=0}^{\infty} \sum_{j=0}^{M-1} \sum_{i=1}^{k} \pi(n, j, i)\ [M\,n + j]2 = M2n=1\infty nn-1\pi ne+n=0\infty n\pi ne+n=0\infty \pi n\eta$ +2M$n=0\infty n\ \pi n\xi$

$=M^2[\Phi''(1)e + \sum_{i=0}^{m*-1} i\pi_i\, e + \pi_{m*} m^*(I-R)^{-1}e + \pi_{m*}(I-R)^{-2}$ Re] $+ \sum_{i=0}^{m*-1} \pi_i \eta + \pi_{m*}(I-R)^{-1}\eta + 2M[\sum_{i=0}^{m*-1} i\pi_i\, \xi + \pi_{m*} m^*(I-R)^{-1}\xi + \pi_{m*}(I-R)^{-2}R\xi]$. This gives E $(S^2) = M^2[\ \sum_{i=1}^{m*-1} i(i-1)\pi_i\, e\ + m^*(m^*-1)\pi_{m*}\ (I - R)^{-1} e\ + 2m^*\pi_{m*}\ (I-R)^{-2}Re + 2\pi_{m*}(I-R)^{-3}\ R^2\ e + \sum_{i=0}^{m*-1} i\pi_i\, e + \pi_{m*} m^*(I-R)^{-1}e\ + \pi_{m*}(I-R)^{-2}$ Re] $+ \sum_{i=0}^{m*-1} \pi_i \eta + \pi_{m*}(I-R)^{-1}\eta + 2M\ [\ \sum_{i=0}^{m*-1} i\pi_i\, \xi + \pi_{m*} m^*(I-R)^{-1}\xi + \ \pi_{m*}(I-R)^{-2}\ R\ \xi]$. (43)

Using (41) and (43) the variance can be written for Sub Case (A3).

## III. MODEL (B). MAXIMUM ARRIVAL SIZE M IS LESS THAN MAXIMUM SERVICE SIZE N

In this Model (B) the dual case of Model (A), namely the case, M < N is treated. Here the partitioning matrices are of order Nk and the customers are considered as members of N blocks, M plays no role in the partition where as it played the major role in Model (A). Two Sub Cases namely (B1) N ≥ C and (B2) C > N come up in the Model (B). (When M =N and for various values of C greater than them, or less than them or equal to them, both models are applicable and one can use any one of them.) The assumption (v) of Model (A) alone is modified without changing other assumptions stated earlier for the same.

**3.1Assumption.**

v) The maximum of the maximums of arrival sizes in all the environments M = $\max_{1 \le i \le k} M_i$ is less than the maximum of the maximums of service sizes in all the environments N = $\max_{1 \le i \le k} N_i$ where the maximum arrival and service sizes are $M_i$ and $N_i$ in the environment i for 1 ≤ i ≤ k.

**3.2Analysis**

Since this model is dual, the analysis is similar to that of Model (A). The differences are noted below. The state space of the chain is as follows defined in a similar way presented for Model (A).

X (t) = {(n, j, i): for 0 ≤ j ≤ N-1 for 1 ≤ i ≤ k and 0 ≤ n < ∞} (44)

The chain is in the state (n, j, i) when the number of customers in the queue is, n N + j, and the environment state is i for 0 ≤ j ≤ N-1, for 1 ≤ i ≤ k and 0 ≤ n < ∞. When the customers in the system is r then r is identified with (n, j) where r on division by N gives n as the quotient and j as the remainder.

**3.2.1 Sub Case: (B1) N ≥ C**

The infinitesimal generator $Q_{B,3.1}$ of the Sub Case (B1) of Model (B) has the same block partitioned structure given in (4) for the Sub Cases (A1) and (A2) of Model (A) but the inner matrices are of different orders and elements.

$$Q_{B,3.1}=\begin{bmatrix} B''_1 & A''_0 & 0 & 0 & . & . & . & \cdots \\ A''_2 & A''_1 & A''_0 & 0 & . & . & . & \cdots \\ 0 & A''_2 & A''_1 & A''_0 & 0 & . & . & \cdots \\ 0 & 0 & A''_2 & A''_1 & A''_0 & 0 & . & \cdots \\ 0 & 0 & 0 & A''_2 & A''_1 & A''_0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \qquad (45)$$

In (45) the states of the matrices are listed lexicographically as $\underline{0}, \underline{1}, \underline{2}, \underline{3}, \ldots. \underline{n}$, ....Here the state vector is given as follows. $\underline{n}$ = ((n, 0, 1),…(n, 0, k),(n, 1, 1),…(n, 1, k),(n, 2, 1),…(n, 2, k),…(n, N-1, 1),…(n, N-1, k)), for 0 ≤ n < ∞. The matrices, $B''_1,\ A''_0\ , A''_1\ and\ A''_2$ are all of order Nk. The matrices $B''_1\ and\ A''_1$ have negative diagonal elements and their off diagonal elements are non- negative. The matrices $A''_0\ and\ A''_2$ have nonnegative elements. They are all given below. As in model (A), letting $\Lambda_j\, , for\ 1\ \le j\ \le M\ , and\ \ U_j\, , V_j\ \ for\ 1\ \le j\ \le N$, Λ and U as diagonal matrices of order k given by (5) and (6) and letting $Q_1^{'} = Q_1 - \Lambda - CU$, the partitioning matrices are defined below

$$A''_0 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \Lambda_M & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \Lambda_{M-1} & \Lambda_M & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Lambda_2 & \Lambda_3 & \cdots & \Lambda_M & 0 & 0 & \cdots & 0 \\ \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-1} & \Lambda_M & 0 & \cdots & 0 \end{bmatrix} (46) \qquad A''_2 \begin{bmatrix} CU_N & CU_{N-1} & CU_{N-2} & \cdots & CU_2 & CU_1 \\ 0 & CU_N & CU_{N-1} & \cdots & CU_3 & CU_2 \\ 0 & 0 & CU_N & \cdots & CU_4 & CU_3 \\ 0 & 0 & 0 & \ddots & CU_5 & CU_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & CU_{N-1} & CU_{N-2} \\ 0 & 0 & 0 & \cdots & CU_N & CU_{N-1} \\ 0 & 0 & 0 & \cdots & 0 & CU_N \end{bmatrix} (47)$$

$$A''_1 = \begin{bmatrix} Q_1^{'} & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_M & 0 & 0 & \cdots & 0 & 0 \\ CU_1 & Q_1^{'} & \Lambda_1 & \cdots & \Lambda_{M-1} & \Lambda_M & 0 & \cdots & 0 & 0 \\ CU_2 & CU_1 & Q_1^{'} & \cdots & \Lambda_{M-2} & \Lambda_{M-1} & \Lambda_M & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ CU_{N-M-1} & CU_{N-M-2} & CU_{N-M-3} & \cdots & Q_1^{'} & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-1} & \Lambda_M \\ CU_{N-M} & CU_{N-M-1} & CU_{N-M-2} & \cdots & CU_1 & Q_1^{'} & \Lambda_1 & \cdots & \Lambda_{M-2} & \Lambda_{M-1} \\ CU_{N-M+1} & CU_{N-M} & CU_{N-M-1} & \cdots & CU_2 & CU_1 & Q_1^{'} & \cdots & \Lambda_{M-3} & \Lambda_{M-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ CU_{N-2} & CU_{N-3} & CU_{N-4} & \cdots & CU_{N-M-2} & CU_{N-M-3} & CU_{N-M-2} & \cdots & Q_1^{'} & \Lambda_1 \\ CU_{N-1} & CU_{N-2} & CU_{N-3} & \cdots & CU_{N-M-1} & CU_{N-M-2} & CU_{N-M-1} & \cdots & CU_1 & Q_1^{'} \end{bmatrix} (48)$$

$$B''_1=\begin{bmatrix} Q'_{1,0} & \Lambda_1 & \Lambda_2 & \cdots & . & \cdots & \Lambda_M & 0 & 0 & \cdots & . & . & \cdots & 0 & 0 \\ U & Q'_{1,1} & \Lambda_1 & \cdots & . & \cdots & \Lambda_{M-1} & \Lambda_M & 0 & \cdots & . & . & \cdots & 0 & 0 \\ 2V_1 & 2U_1 & Q'_{1,2} & \cdots & . & \cdots & \Lambda_{M-2} & \Lambda_{M-1} & \Lambda_M & \cdots & . & . & \cdots & 0 & 0 \\ 3V_2 & 3U_2 & 3U_1 & \cdots & . & \cdots & . & . & . & \cdots & . & . & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ CV_{C-1} & CU_{C-1} & CU_{C-2} & \cdots & Q'_1 & \cdots & . & . & . & \cdots & \Lambda_M & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ CV_{N-M-2} & CU_{N-M-2} & CU_{N-M-3} & \cdots & . & \cdots & Q'_1 & \Lambda_1 & \Lambda_2 & \cdots & . & . & \cdots & \Lambda_{M-1} & \Lambda_M \\ CV_{N-M-1} & CU_{N-M-1} & CU_{N-M-2} & \cdots & . & \cdots & CU_1 & Q'_1 & \Lambda_1 & \cdots & . & . & \cdots & \Lambda_{M-2} & \Lambda_{M-1} \\ \vdots & \vdots & \vdots & \ddots & . & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ CV_{N-3} & CU_{N-3} & CU_{N-4} & \cdots & . & \cdots & CU_{N-M-2} & CU_{N-M-3} & CU_{N-M-2} & \cdots & . & . & \cdots & Q'_1 & \Lambda_1 \\ CV_{N-2} & CU_{N-2} & CU_{N-3} & \cdots & . & \cdots & CU_{N-M-1} & CU_{N-M-2} & CU_{N-M-1} & \cdots & . & . & \cdots & CU_1 & Q'_1 \end{bmatrix} \quad (49)$$

Here, $Q'_{1,j} = Q_1 - \Lambda - jU$ for $0 \le j \le C$ and $Q'_{1,C} = Q'_1$. In (49) the case C > N has been presented. When C=N, $V_j$ and $U_j$ in $B''_1$ do not get C as multiplier in (49) and C appears as a multiplier of $U_j$ in $A''_2$ and $A''_1$ in (47) and (48). The multiplier of matrices $U_j$ $and$ $V_j$ concerning the services increase by one in each row block from third row block as the row number increases by one, up to the row C+1 and it remains C in row blocks after that as given above.

$$Q''_B=$$

$$\begin{bmatrix} Q'_1+CU_N & \Lambda_1+CU_{N-1} & \cdots & \Lambda_{M-1}+CU_{N-M+1} & \Lambda_M+CU_{N-M} & CU_{N-M-1} & \cdots & CU_2 & CU_1 \\ CU_1 & Q'_1+CU_N & \cdots & \Lambda_{M-2}+CU_{N-M+2} & \Lambda_{M-1}+CU_{N-M+1} & \Lambda_M+CU_{N-M} & \cdots & CU_3 & CU_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ CU_{N-M-2} & CU_{N-M-3} & \cdots & Q'_1+CU_N & \Lambda_1+cU_{N-1} & \Lambda_2+CU_{N-2} & \cdots & \Lambda_M+CU_{N-M} & CU_{N-M-1} \\ CU_{N-M-1} & CU_{N-M-2} & \cdots & CU_1 & Q'_1+CU_N & \Lambda_1+CU_{N-1} & \cdots & \Lambda_{M-1}+CU_{N-M+1} & \Lambda_M+CU_{N-M} \\ \Lambda_M+CU_{N-M} & CU_{N-M-1} & \cdots & CU_2 & CU_1 & Q'_1+CU_N & \cdots & \Lambda_{M-2}+CU_{N-M+2} & \Lambda_{M-1}+CU_{N-M+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Lambda_2+CU_{N-2} & \Lambda_3+CU_{N-3} & \cdots & CU_{N-M-1} & CU_{N-M-2} & CU_{N-M-3} & \cdots & Q'_1+CU_N & \Lambda_1+CU_{N-1} \\ \Lambda_1+CU_{N-1} & \Lambda_2+CU_{N-2} & \cdots & \Lambda_M+CU_{N-M} & CU_{N-M-1} & CU_{N-M-2} & \cdots & CU_1 & Q'_1+CU_N \end{bmatrix} \quad (50)$$

The basic generator (50) which is concerned with only the arrival and service is $Q''_B = A''_0 + A''_1 + A''_2$. This is also block circulant. Using similar arguments given for Model (A) it can be seen that its probability vector is $w = \left(\frac{\phi}{N}, \frac{\phi}{N}, \frac{\phi}{N}, \dots, \frac{\phi}{N}\right)$ and the stability condition remains the same. Following the arguments given for Sub Cases (A1) and (A2) of Model (A), one can find the stationary probability vector for Sub Case (B1) of Model (B) also in matrix geometric form. All performance measures in section 2.3 including the expectation of customers waiting for service and its variance for Sub Cases (A1) and (A2) of Model (A) are valid for Sub Case (B1) of Model (B) with M is replaced by N. It can also be seen that when N = C the system admits Matrix Geometric solution as in Model (A).

### 3.2.2Sub Case: (B2) C > N

The infinitesimal generator $Q_{B,3.2}$ of the Sub Case (B2) of Model (B) has the same block partitioned structure given in (25) for Sub Case (A3) of Model (A) but the inner matrices are of different orders and elements. When C > N > M, the M/M/C bulk queue admits a modified matrix geometric solution as follows. The chain X (t) describing this Sub Case (B2), can be defined as in the Sub Case (B1). It has the infinitesimal generator $Q_{B,3.2}$ of infinite order which can be presented in block partitioned form given below. When C > N, let C = m* N + n* where m* is positive integer and n* is nonnegative integer with 0 ≤ n* ≤ N-1.

$$Q_{B,3.2}=\begin{bmatrix} B'''_1 & A''_0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ A''_{2,1} & A''_{1,1} & A''_0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ 0 & A''_{2,2} & A''_{1,2} & A''_0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & A''_{2,3} & A''_{1,3} & A''_0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & A''_{2,m*} & A''_{1,m*} & A''_0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & A''_2 & A''_1 & A''_0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & A''_2 & A''_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \vdots & \ddots \end{bmatrix} \quad (51)$$

In (51) the states of the matrices are listed lexicographically as $\underline{0}, \underline{1}, \underline{2}, \underline{3}, \dots \underline{n}, \dots$. Here the vector $\underline{n}$ is of type 1 x k N and $\underline{n}$ = ((n, 0, 1),(n, 0, 2)…(n, 0, k),(n, 1, 1),(n, 1, 2)…(n, 1, k)…(n, N-1, 1),(n, N-1, 2)…(n, N-1, k) for n ≥ 0. The matrices $B'''_1$, $A''_{1j}$ for 1 ≤ j ≤ m* and $A''_1$ have negative diagonal elements, they are of order Nk and their off diagonal elements are non- negative. The matrices $A''_0, A''_{2,j}$ *and* $A''_2$ for 1 ≤ j ≤ m* have nonnegative elements and are of order Nk and the matrices $A''_0, A''_1$ *and* $A''_2$ are same as defined earlier for Sub Case (B1) in equations (46), (47) and (48). Since C > N the number of servers in the system s equals the number of customers in the system L up to customer length becomes C= m* N + n*. When the number of customers becomes more than C, (L ≥ C), the number of servers in the system becomes constant C. When the number of customers becomes less than C, the number of servers again falls and equals the number of customers. As in model (A), letting $\Lambda_j, for\ 1 \le j \le M, and\ U_j, V_j\ for\ 1 \le j \le N$, Λ and U as diagonal matrices of order k given by (5) and (6) and letting $Q_1' = Q_1 - \Lambda - CU$, the partitioning matrices are defined as follows. The matrix $A''_{2,j}$ is given below for 1 ≤ j < m*-1 is given below.

$$A''_{2,j} = \begin{bmatrix} jNU_N & jNU_{N-1} & \cdots & jNU_2 & jNU_1 \\ 0 & (jN+1)U_N & \cdots & (jN+1)U_3 & (jN+1)U_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & (jN+N-2)U_N & (jN+N-2)U_{N-1} \\ 0 & 0 & \cdots & 0 & (jN+N-1)U_N \end{bmatrix} \quad (52)$$

The matrix $A_{2,m*}$ is as follows given in (53) when C = m*N + n* is such that 0 ≤ n* ≤ N-1. $A''_{2,m*} =$

$$\begin{bmatrix} (Nm*)U_N & (Nm*)U_{N-1} & \cdots & . & \cdots & (Nm*)U_2 & (Nm*)U_1 \\ 0 & (Nm*+1)U_N & \cdots & . & \cdots & (Nm*+1)U_3 & (Nm*+1)U_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & CU_N & \cdots & CU_{n*+2} & CU_{n*+1} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & CU_N & CU_{N-1} \\ 0 & 0 & \cdots & 0 & \cdots & 0 & CU_N \end{bmatrix} \quad (53)$$

Considering again, $Q'_{1,j} = Q_1 - \Lambda - jU$ for 0 ≤ j ≤ C and $Q'_{1,C} = Q'_1$ as in Sub Cases (A1) and(A2),

$B'''_1 =$

$$\begin{bmatrix} Q'_{1,0} & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_M & 0 & \cdots & 0 & 0 \\ U & Q'_{1,1} & \Lambda_1 & \cdots & \Lambda_{M-1} & \Lambda_M & \cdots & 0 & 0 \\ 2V_1 & 2U_1 & Q'_{1,2} & \cdots & \Lambda_{M-2} & \Lambda_{M-1} & \cdots & 0 & 0 \\ 3V_2 & 3U_2 & 3U_1 & \cdots & . & . & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (N-M-1)V_{N-M-2} & (N-M-1)U_{N-M-2} & (N-M-1)U_{N-M-3} & \cdots & Q'_{1,N-M-1} & \Lambda_1 & \cdots & \Lambda_{M-1} & \Lambda_M \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (N-2)V_{N-3} & (N-2)U_{N-3} & (N-2)U_{N-4} & \cdots & (N-2)U_{N-M-2} & (N-2)U_{N-M-3} & \cdots & Q'_{1,N-2} & \Lambda_1 \\ (N-1)V_{N-2} & (N-1)U_{N-2} & (N-1)U_{N-3} & \cdots & (N-1)U_{N-M-1} & (N-1)U_{N-M-2} & \cdots & (N-1)U_1 & Q'_{1,N-1} \end{bmatrix} \quad (54)$$

The matrix $A''_{1,j}$ for 1 ≤ j ≤ m*-1 is as follows.

$A''_{1,j}$

$$= \begin{bmatrix} Q'_{1,jN} & \Lambda_1 & \cdots & \Lambda_M & 0 & \cdots & 0 & 0 \\ (jN+1)U_1 & Q'_{1,jN+1} & \cdots & \Lambda_{M-1} & \Lambda_M & \cdots & 0 & 0 \\ (jN+2)U_2 & (jN+2)U_1 & \cdots & \Lambda_{M-2} & \Lambda_{M-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (jN+N-M-1)U_{N-M-1} & (jN+N-M-1)U_{N-M-2} & \cdots & Q'_{1,jN+N-M-1} & \Lambda_1 & \cdots & \Lambda_{M-1} & \Lambda_M \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (jN+N-2)U_{N-2} & (jN+N-2)U_{N-3} & \cdots & (jN+N-2)U_{N-2-M} & (jN+N-2)U_{N-3-M} & \cdots & Q'_{1,jN+N-2} & \Lambda_1 \\ (jN+N-1)U_{N-1} & (jN+N-1)U_{N-2} & \cdots & (jN+N-1)U_{N-1-M} & (jM+M-1)U_{N-2-M} & \cdots & (jM+M-1)U_1 & Q'_{1,jN+N-1} \end{bmatrix} \quad (55)$$

The matrix $A''_{1,m*}$ is in (56) when C = m*N + n* and 0 ≤ n* ≤ N-1. From row block n*+1, the multiplier of $U_j$ is

C. The matrix $A''_{1,m*} = \begin{bmatrix} Q'_{1,Nm*} & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_M & 0 & 0 & \cdots & 0 & 0 \\ (Nm*+1)U_1 & Q'_{1,Nm*+1} & \Lambda_1 & \cdots & \Lambda_{M-1} & \Lambda_M & 0 & \cdots & 0 & 0 \\ (Nm*+2)U_2 & (Nm*+2)U_1 & Q'_{1,Nm*+2} & \cdots & \Lambda_{M-2} & \Lambda_{M-1} & \Lambda_M & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ CU_{n*} & CU_{n*-1} & CU_{n*-2} & \cdots & Q'_{1,C} & \Lambda_1 & \Lambda_2 & \cdots & . & . \\ CU_{n*+1} & CU_{n*} & CU_{n*-1} & \cdots & CU_1 & Q'_1 & \Lambda_1 & \cdots & . & . \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ CU_{N-2} & CU_{N-3} & CU_{N-4} & \cdots & CU_{N-M-2} & CU_{N-M-3} & CU_{N-M-2} & \cdots & Q'_1 & \Lambda_1 \\ CU_{N-1} & CU_{N-2} & CU_{N-3} & \cdots & CU_{N-M-1} & CU_{N-M-2} & CU_{N-M-1} & \cdots & CU_1 & Q'_1 \end{bmatrix} \quad (56)$

The basic generator for this model is also same as (50) which is concerned with only the arrival and the service. $Q''_B = A''_0 + A''_1 + A''_2$. This is also block circulant. Using similar arguments given for Model (A) it can be seen that its probability vector is $w = \left(\frac{\phi}{N}, \frac{\phi}{N}, \frac{\phi}{N}, \dots, \frac{\phi}{N}\right)$ and the stability condition remains the same. Following the arguments given for Sub Case (A3) in section 2.2.2 of Model (A), one can find the stationary probability vector for Sub Case (B2) of Model (B) also in modified matrix geometric form. All the performance measures given in section 2.3 including the expectation of customers waiting for service and its variance for Sub Case (A3) are valid for Sub Case (B2) of Model (B) except M is replaced by N.

## IV. NUMERICAL ILLUSTRATION

For the models the varying environment is considered to be governed by the Matrix $Q_1 = \begin{bmatrix} -5 & 2 & 3 \\ 2 & -4 & 2 \\ 4 & 2 & -6 \end{bmatrix}$. The arrival time and service time parameters of exponential distributions are respectively fixed in the three environments E1, E2 and E3 as $\underline{\lambda} = (10, 12, 14)$ $and$ $\underline{\mu} = (4, 5, 6)$. Twelve examples are studied with various values for C, M and N. The maximum arrival size M and the maximum service size N in all environments are (i) M=N=4 in four examples, (ii) M=4, N=3 in four examples studied for Model (A) and (iii) M=3, N=4 in four examples studied for Model (B). In all these sets of different values of M and N, mentioned in (i), (ii) and (iii) the number of servers C are varied as C = 3, 4, 6 and 7. When M=N=4 the probabilities of bulk arrival and bulk service sizes in the environments are as follows given in Table1.

**Table 1: Probabilities of χ Arrival and ψ Service Sizes with Maximums = 4 in Three Environments.**

| Environment | P(Arrival size =1) | P(Arrival size= 2) | P(Arrival size =3) | P(Arrival size =4) | P(service size =1) | P(service size =2) | P(Servi ce size =3) | P(Servic e size =2) |
|---|---|---|---|---|---|---|---|---|
| E1 | .5 | .4 | 0 | .1 | .5 | .3 | .1 | .1 |
| E2 | .6 | .4 | 0 | 0 | .7 | .2 | .1 | 0 |
| E3 | .7 | .3 | 0 | 0 | .8 | .2 | 0 | 0 |

For the cases with M=3, without changing the arrival size probabilities given above in Table 1 for the environments E2 and E3, the arrival size probabilities in E1 are changed as follows P(Arrival size =1 ) =.5 P(Arrival size =2) =.4, P(Arrival size=3)= .1 and P(Arrival size =4) =0. For the cases with N=3, without changing the service size probabilities given above in Table 1 for the environments E2 and E3, the service size probabilities in E1 are changed as follows P(Service size =1 ) =.5 P(Service size =2) =.3, P(Service size =3)= .2 and P(Service size = 4) =0. Here same numbers of 30 iterations are performed to find the rate matrix R in all the twelve cases. When C =3 and C =4 matrix geometric results are seen and the results obtained are presented in Table2. When C= 6 and C=7 modified matrix geometric results are seen and the results obtained are presented in Table 3. Probabilities of various sizes of queue lengths S = 0, 1, 2, 3 and various blocks 0, 1, 2, 3 are obtained. Further P(S> 15), E(S) and VAR (S) are also derived for all the twelve cases. Norms, arrival rate and service rate values are presented. Figure 1 and figure 2 present probabilities of various queue sizes and block sizes for the twelve examples. Effects of variation of rates on expected queue length and on probabilities of queue lengths are exhibited in tables 2 and 3 respectively. The decrease in arrival rates (so also increase in service rates) makes the convergence of R matrix faster which can be seen in the decrease of norm values.

**Table2: Results Obtained For Matrix Geometric Models.**

| | C=3,M=4=N | C=3,M=4,N=3 | C=3,M=3,N=4 | C=4=M=N | c=4,M=4,N=3 | C=4,M=3,N=4 |
|---|---|---|---|---|---|---|
| **P(S=0)** | 0.079684166 | 0.072645296 | 0.086694481 | 0.134207116 | 0.127898311 | 0.138717179 |
| **P(S=1)** | 0.094816229 | 0.089299727 | 0.103026395 | 0.160159184 | 0.158139375 | 0.165224098 |
| **P(S=2)** | 0.099972009 | 0.094739563 | 0.10844421 | 0.168932101 | 0.167596052 | 0.173956131 |
| **P(S=3)** | 0.080822795 | 0.077354801 | 0.088540336 | 0.1362632 | 0.136279652 | 0.141667691 |
| **$\pi$0e** | 0.3552952 | 0.334039388 | 0.386705422 | 0.599561601 | 0.589913389 | 0.619565098 |
| **$\pi$1e** | 0.246062643 | 0.239774491 | 0.258608447 | 0.269420604 | 0.27246095 | 0.268090953 |
| **$\pi$2e** | 0.151711966 | 0.152962345 | 0.149421908 | 0.087788452 | 0.091032247 | 0.079174934 |

| | | | | | | |
|---|---|---|---|---|---|---|
| **π3e** | 0.093923215 | 0.098003396 | 0.086467581 | 0.028918419 | 0.030765921 | 0.023368742 |
| **P(S>15)** | 0.153006976 | 0.175220381 | 0.118796643 | 0.014310924 | 0.015827493 | 0.009800274 |
| **Norm** | 8.607020E-05 | 0.000121229 | 5.72322E-05 | 1.70312E-07 | 2.66577E-07 | 6.89660E-08 |
| **Arrival Rate** | 4.322222222 | 4.322222222 | 4.22962963 | 4.322222222 | 4.322222222 | 4.22962963 |
| **Service rate** | 5.35 | 5.238888889 | 5.35 | 7.133333333 | 6.985185185 | 7.133333333 |
| **E(S)** | 8.177499726 | 8.832269818 | 7.221716529 | 3.754933893 | 3.853312418 | 3.52779164 |
| **Var(S)** | 69.84639946 | 81.04471283 | 53.74109494 | 13.38646744 | 13.99187868 | 11.37540035 |

**Table3: Results Obtained For Modified Matrix Geometric Models**

| | **C=6,M=N=4** | **C=6,M=4,N=3** | **C=6,M=3,N=4** | **C=7,M=N=4** | **C=7,M=4,N=3** | **C=6,M=3,N=4** |
|---|---|---|---|---|---|---|
| **P(S=0)** | 0.158294126 | 0.152216922 | 0.160879886 | 0.16058125 | 0.15450255 | 0.16284145 |
| **P(S=1)** | 0.189049227 | 0.18816547 | 0.191805374 | 0.19176482 | 0.1909816 | 0.19413942 |
| **P(S=2)** | 0.19939206 | 0.199573572 | 0.20194472 | 0.20224439 | 0.20255169 | 0.20439655 |
| **P(S=3)** | 0.160751929 | 0.162182332 | 0.164403097 | 0.16305786 | 0.16458877 | 0.16641004 |
| **π0e** | 0.707487342 | 0.702138296 | 0.719033078 | 0.71764832 | 0.71262461 | 0.72778746 |
| **π1e** | 0.253070365 | 0.25695515 | 0.248707867 | 0.2598637 | 0.25727703 | 0.24859244 |
| **π2e** | 0.03378797 | 0.034926405 | 0.028523013 | 0.02598637 | 0.02682887 | 0.02168998 |
| **π3e** | 0.004819942 | 0.005080227 | 0.003299621 | 0.0027537 | 0.00289556 | 0.00176931 |
| **P(S>15)** | 0.000834381 | 0.000899922 | 0.000436421 | 0.00034835 | 0.00037393 | 0.0001608 |
| **Norm** | 3.83737E-12 | 5.86712E-12 | 7.50084E-13 | 5.06990E-14 | 7.6660E-14 | 7.2997E-15 |
| **Arrival Rate** | 4.322222222 | 4.322222222 | 4.22962963 | 4.322222222 | 4.322222222 | 4.22962963 |
| **Service rate** | 10.7 | 10.47777778 | 10.7 | 12.48333333 | 12.2240741 | 12.4833333 |
| **E(S)** | 3.687861529 | 3.728565034 | 3.624089963 | 3.57175104 | 3.60833246 | 3.52397251 |
| **Var(S)** | 6.669071323 | 6.652556207 | 6.393367815 | 6.24450839 | 6.21272314 | 6.06130732 |

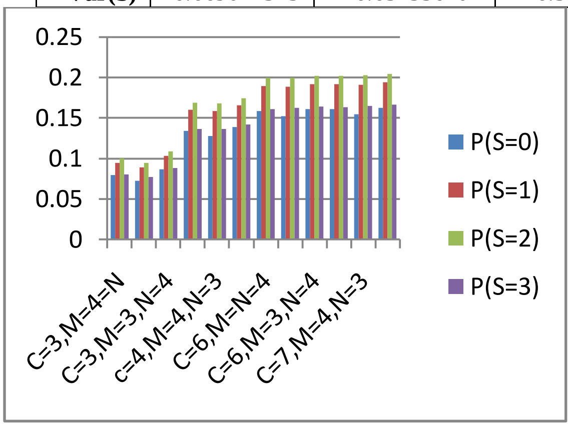


**Figure 1: Probabilities of Queue lengths**

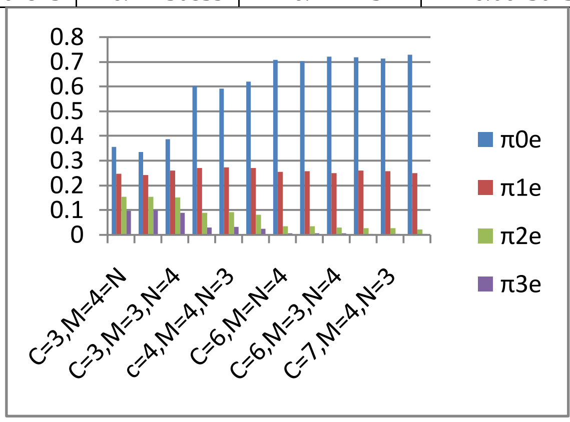


**Figure 2: Probabilities of Block Queue Sizes**

## V. CONCLUSION

Two M/M/C bulk arrival and bulk service queues and their sub cases with randomly varying environments have been treated. The environment changes the arrival rates, the service rates, and the probabilities of sizes of bulk arrivals and bulk services. Matrix geometric and modified matrix geometric results have been obtained by suitably partitioning the infinitesimal generator by grouping of customers and environments together respectively when the number of servers is not greater than or greater than the maximum of the maximum arrival and maximum service sizes. The basic system generators of the queues are block circulant matrices which are explicitly presenting the stability condition in standard form. Numerical results for various bulk queue models are presented and discussed. Effects of variation of rates on expected queue length and on probabilities of queue lengths are exhibited. The decrease in arrival rates (so also increase in service rates) makes the convergence of R matrix faster which can be seen in the decrease of norm values. The variances also decrease. Bulk PH/PH/C queue with randomly varying environments causing changes in sizes of the PH phases may produce further results if studied since PH/PH/C queue is a most general form almost equivalent to G/G/C queue.

## VI. ACKNOWLEDGEMENT

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